Math 429 - Exercise Sheet 14

1. Compute the character of the representation $\wedge^k \mathbb{C}^n$ of $\mathfrak{sl}_n$, given our explicit description of its weight spaces in Lecture 13, and use this to verify the Weyl character formula (197).

Solution. Let $(v_1, \ldots, v_n)$ be a basis of $\mathbb{C}^n$. Then every element $v_{i_1} \wedge \cdots \wedge v_{i_k}$, $1 \le i_1 < \cdots < i_k \le n$ of the induced basis of $\wedge^i \mathbb{C}^n$ is a weight vector. More precisely, the diagonal matrix $(x_1, \ldots, x_n)$ in the Cartan subalgebra $\mathfrak{h}$ acts as

$$(x_1, \ldots, x_n) v_{i_1} \wedge \cdots \wedge v_{i_k} = (x_{i_1} + \cdots + i_k) v_{i_1} \wedge \cdots \wedge v_{i_k}.$$

These eigenvalues correspond to the elements

$$e_{i_1} + \cdots + e_{i_k}, \quad 1 \le i_1 < \cdots < i_k \le n$$

in the weight lattice, and the highest weight is $\omega_k = e_1 + \cdots + e_k$. Setting $z_i = \exp(e_i)$, the above decomposition tells us that the character of this representation is the elementary symmetric polynomial

$$\chi(\omega_k) = \sum_{1 \le i_1 < \cdots < i_k \le n} z_{i_1} \cdots z_{i_k}. \tag{1}$$

We verify that(1) corresponds to Weyl's formula. The symbol e^{ω_k} corresponds to the monomial $z_1 \ldots z_k$. Plugging this inside the expression in paragraph 14.5 of the Lecture Notes we get

$$\chi(\omega_k) = \frac{\sum_{w \in S_n} \operatorname{sgn}(w) \prod_{i=1}^n z_{w(i)}^{\delta_{i \le k} + n - i}}{\prod_{1 \le i < j \le n} (z_i - z_j)}, \tag{2}$$

where $\delta_{i \le k}$ is the delta function for the condition $i \le k$. First we observe that the expression (2) is symmetric. In fact, permuting the variables via an element $w' \in S_n$ changes both the numerator and the denominator by a factor of $\operatorname{sgn}(w')$. Since any symmetric function has poles of even degree[1], we deduce that the simple singularities at $z_i = z_j$ in (2) must be solvable. In other words the denominator divides the numerator, and the whole expression is a polynomial in the z's. Finally, we use the following minimizing property of the elementary symmetric function (1), which is easy to check.

Lemma 1. *Let $f(z_1, \ldots, z_n)$ be a symmetric function. If the fraction $\frac{f(tz_1, \ldots, tz_k, z_{k+1}, \ldots, z_n)}{t^k}$ has finite limit for $t \to \infty$, then f equals the elementary function (1)[2].*

We compute the highest power of t occurring in the numerator and denominator of (2) when scaling the variables as in Lemma 1. In the denominator, such highest power occurs in

$$(tz_1)^n (tz_2)^{n-1} \ldots (tz_k)^{n-k+1} z_{k+1}^{n-k-1} \ldots z_{n-1} = t^{k(n - \frac{k-1}{2})} z_1^n z_2^{n-1} \ldots z_k^{n-k+1} z_{k+1}^{n-k-1} \ldots z_{n-1}. \tag{3}$$

[1] You can prove this by expanding any symmetric function $f(z_1, \ldots, z_n)$ at $|z_i - z_j| << 1$, while considering the other variables as constants.

[2] The symmetry property implies that this condition has to hold when scaling whichever k variables.

On the other hand, the denominator becomes

$$\prod_{1\le i<j\le k} t(z_i - z_j) \prod_{1\le i<\le k<j\le n} (tz_i - z_j) \prod_{k<i<j\le n} (z_i - z_j).$$

The first factor gives $t^{k(k+1)/2}$. In the second factor every index $1 \le i < j \le k$ appears $n-k-1$ times, so the exponents add up to

$$t^{k(k+1)/2} t^{k(n-k-1)}. \tag{4}$$

Multiplying the inverse of (4) with (3), we find exactly t^k.

2. Looking back to Exercise 5 on last week's sheet, compute the character of the tautological representation of $\mathfrak{o}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{o}_{2n}$, respectively (and use this to verify the Weyl character formula).

Solution. We solve the exercise explicitely in type D_n, the other ones being analogous. Remember our choice of Cartan subalgebra $\mathfrak{h} \subset \mathfrak{o}_{2n}$, consisting of block-diaginal matrices, having the matrix

$$\begin{bmatrix} 0 & a_k \\ -a_k & 0 \end{bmatrix},\ a_k \in \mathbb{C}$$

in the kth block. It is then clear that the weight decomposition of $\mathbb{C}^{2n}$ for the tautological action of $\mathfrak{o}_{2n}$ is given by the one dimensional weight spaces generated by the vectors

$$v_k^{\pm} = (0 \dots 0 \overbrace{\ 1\ }^{k\text{th pos.}} \pm i\ 0 \dots 0)^T.$$

Furthermore, the eigenvalue associated to $v_k^{\pm}$ sends a matrix in the Cartan subalgebra to $\pm i a_k$. Recalling our description of the root system in Sheet 9, we see that this corresponds to the element $\pm e_k$ in the weight lattice. We can write down the character of this representation as a Laurent polynimial in the variables

$$z_k = \exp(e_k).$$

According to the above analysis, we get

$$\chi(e_1) = \sum_{i=1}^{n} \left(z_i + \frac{1}{z_i} \right). \tag{5}$$

We observe that the last expression is symmetric with respect to the usual S_n action, and with respect to the reflections $z_i \mapsto 1/z_i$, corresponding to $e_i \mapsto -e_i$. These trasformations actually generate the whole Weyl group of the D_n root system (which you can think inside S_{2n}).
We check that the expression of Weyl character formula has the same symmetries as (5) by expressing it in the z variables. The sum ρ of all positive roots is

$$\frac{1}{2} \sum_{1\le i<j\le n} (e_i - e_j) + (e_i + e_j) = \sum_{i=1}^{n-1} (n-i) e_i,$$

so that $\exp(\rho) = z_1^{n-1} z_2^{n-2} \dots z_{n-1}$. Since the highest weight of this representation is e_1, the numerator of Weyl's formula is

$$\sum_{w\in W} \mathrm{sgn(w)}\, z_{w(1)}^{n} z_{w(2)}^{n-2} z_{w(3)}^{n-3} \cdots z_{w(n-1)}. \tag{6}$$

Assuming the above description of the Weyl group, we see that permuting the variables and applying reflections $z_i \mapsto 1/z_i$ changes the expression (6) by the sign of the transformation. The denominator becomes

$$\prod_{1\le i<j\le n} \left(\frac{z_i}{z_j} - \frac{z_j}{z_i}\right)\left(z_i z_j - \frac{1}{z_i z_j}\right), \tag{7}$$

which has clearly the same behavior as the numerator with respect to the above transformations. Thus the product

$$\sum_{w\in W} \mathrm{sgn(w)}\, z_{w(1)}^{n} z_{w(2)}^{n-2} z_{w(3)}^{n-3} \cdots z_{w(n-1)} \cdot \prod_{1\le i<j\le n} \left(\frac{z_i}{z_j} - \frac{z_j}{z_i}\right)^{-1}\left(z_i z_j - \frac{1}{z_i z_j}\right)^{-1} \tag{8}$$

enjoys the same symmetries as (5). Weyl's theorem tells us that the two expressions are actually the same.

3. The adjoint representation of any simple Lie algebra is $L(\theta)$, where θ denotes the maximal root (i.e. the unique positive root such that $\theta + \alpha \notin R$ for all $\alpha \in R^+$). Compute the character of the adjoint representation of $\mathfrak{sl}_n$, and verify the Weyl character formula.

Solution. The maximal weight of the adjoint representation is the only root in the dominant Weyl chamber. With the usual choices of positive roots and simple roots, this maximal weight is $\theta = e_1 - e_n$. The weight decomposition coincides with the usual root decomposition of $\mathfrak{sl}_n$, so the character in the z variables takes the form

$$(n-1) + \sum_{1\le i<j\le n} \left(\frac{z_i}{z_j} + \frac{z_j}{z_i}\right).$$

4. It is easy to see that $L(0) = \mathbb{C}$ for any semisimple Lie algebra $\mathfrak{g}$ (construct the action explicitly), and so the Weyl character formula implies the equality

$$\sum_{w\in W} \mathrm{sgn}(w) e^{w(\rho)} = \prod_{\alpha\in R^+} (e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}) \tag{9}$$

Prove this formula directly using the theory of root systems (*Hint: show that both sides of the equation are Weyl group anti-invariant*).

Solution. Let $\alpha_1, \ldots, \alpha_N$ be the positive roots in R. We expand the left hand side as

$$\sum_{w\in W} \mathrm{sgn}(w) e^{\frac{1}{2}(w(\alpha_1)+\cdots+w(\alpha_n))}. \tag{10}$$

We must prove that after expanding the product on the right hand side of (9), and after all the cancallations, the only surviving summands are those in (10). We check that the right hand side of (9) is Weyl group anti-invariant. Let $s_k \in W$ be the simple reflection with respect to some simple root α_k. Then the only positive root which becomes negative after applying s_k is α_k, and

$$s_k\left(\prod_{i=1}^{N}(e^{\frac{\alpha_i}{2}} - e^{-\frac{\alpha_i}{2}})\right) = \prod_{i\neq k}(e^{\frac{\alpha_i}{2}} - e^{-\frac{\alpha_i}{2}})(e^{\frac{-\alpha_k}{2}} - e^{\frac{\alpha_k}{2}}) = -\prod_{i=1}^{N}(e^{\frac{\alpha_i}{2}} - e^{-\frac{\alpha_i}{2}}).$$

Since the Weyl group is generated by the reflections at simple roots, this proves our claim. This symmetry property implies that, after expanding the product on the right hand side of (9), a summand

$$\pm e^{\frac{1}{2}(\pm_1 \alpha_1 \ \dots \ \pm_n \alpha_n)}$$

occurs if and only if

$$\pm \operatorname{sgn}(w) e^{\frac{1}{2}(\pm_1 w(\alpha_1) \ \dots \ \pm_n w(\alpha_n))}$$

occurs as well, for every element $w \in W$. Since the summand $e^{\frac{1}{2}(\alpha_1 + \ \dots \ + \alpha_n)}$ clearly occurs, this concludes the exercise.

(*) Consider the following inner product of characters

$$(f, g) = \frac{1}{|W|} \int f \overline{g} \prod_{\alpha \in R} (e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}) \tag{11}$$

where $\int e^{\lambda} = \delta_{\lambda 0}$ and $\overline{e^{\lambda}} = e^{-\lambda}$. Prove that $(f, g) = (g, f)$ and use the Weyl character formula to show that the characters of irreducible representations are orthogonal with respect to (11).